SNAP Centre Workshop

Exponents and Radicals

Introduction

Exponents are used as a way of representing repeated multiplication.

For instance, when we see 4^3, we know that it is equivalent to $(4)(4)(4)$, which – in turn – is equivalent to 64.

The same is true when variables are raised to an exponent; when we see t^3, we know it is equivalent to $(t)(t)(t)$. However, since we don't know the value of t, the expression can't be fully evaluated.

Product, Quotient, and Zero Power Rules for Exponents

Despite not being able to find a numerical value for the expression mentioned and ones like it, exponents prove to be very useful when manipulating certain types of variable expressions.

Example 1 y^3y^2 $Expand\ and\ simplify.$

Since we don't know the value of y, we aren't able to substitute and evaluate the expression. Despite this, we are still able to expand both terms.

$$= [(y)(y)(y)][(y)(y)]$$

Removing the square brackets, we see that the expression is just the same variable being multiplied over and over again.

$$= (y)(y)(y)(y)(y)$$

At this point, the expression can be easily simplified into a single term.

$$= y^5$$

By expanding each term, then simplifying, we were able to represent the original expression as a single term.

Going through the process of expansion and simplification seen in **Example 1** is not necessary every time we want to simplify an exponential expression. Note that $y^3y^2 = y^{3+2} = y^5$. This is an example of the ***Product Rule for Exponents***.

$$\boldsymbol{a^x a^y = a^{x+y}}$$

It is equally valid when more than two terms sharing the same base are involved:

$$a^x a^y a^p a^q = a^{x+y+p+q}$$

Example 2 $r^2t^3tr^2qrq^5$ $Use\ the\ product\ rule\ to\ simplify.$

Since we're using the product rule, there is no need to expand each term. Our first step is recognizing which terms share the same base.

By observation, we can see that the bases involved are $q, r, and\ t$, and we will organize the terms accordingly.

$$= (qq^5)(r^2r^2r)(t^3t)$$

From here, we can simplify each group to a single term. (Keep in mind that a variable not raised to any exponent explicitly is actually raised to the exponent 1.)

$$= (q^{1+5})(r^{2+2+1})(t^{3+1})$$

$$= (q^6)(r^5)(t^4)$$

$$= q^6r^5t^4$$

Using the product rule, we were able to simplify the entire expression without expanding any of the terms.

When presented with an expression using division rather than multiplication, we are able to use the same expansion and simplification process from **Example 1** to simplify the expression into a single term.

Example 3 $\frac{x^5}{x^3}$ $Expand\ and\ simplify.$

Again, our first step is to expand each term.

$$= \frac{(x)(x)(x)(x)(x)}{(x)(x)(x)}$$

Next, we divide out all terms that cancel to 1.

$$= \frac{(x)(x)\cancel{(x)(x)(x)}}{\cancel{(x)(x)(x)}}$$

$$= (x)(x)(1)(1)(1)$$

$$= (x)(x)$$

Finally, we simplify the expression into a single term.

$$= x^2$$

Notice that $\frac{x^5}{x^3} = x^{5-3} = x^2$.

Much like **Example 1**, **Example 3** illustrates another important exponent rule; the ***Quotient Rule for Exponents***.

$$\boldsymbol{\frac{a^x}{a^y} = a^{x-y}}$$

Combined with the product rule, we can see that – when using the same base – all exponents in the numerator are added, while all exponents in the denominator are subtracted:

$$\frac{a^x a^y}{a^p a^q} = \frac{a^{x+y}}{a^{p+q}} = a^{(x+y)-(p+q)} = a^{x+y-p-q}$$

Note the special case in which exponents in the numerator and denominator match:

$$\frac{a^x}{a^x}$$

Assuming a^m is not equal to 0, we know this expression is going to be equal to 1, as it is made up of a term divided by itself.

$$\frac{a^x}{a^x} = 1$$

We also know – given the quotient rule – that we can write the expression as the base a raised to the difference of exponents in the numerator and denominator.

$$\frac{a^x}{a^x} = a^{x-x} = a^0$$

Combining our two results illustrates the ***Zero Power Rule,*** where any number or variable raised to the exponent 0 is equal to 1:

$$\boldsymbol{a^0 = 1}$$

Negative Exponents

Example 4 $\frac{xyzx^2y^2z^3zx^4}{zx^3y^2xw^2z}$ $\quad$ $Use\ product\ \&\ quotient\ rules\ to\ simplify.$

*Like **Example 2**, our first step is to identify what the base terms are. Examining the numerator, it is clear that* $x, y, and\ z$ *are base terms. Be careful, however, to note the* w^2 *in the denominator before proceeding.*

$$= (\frac{1}{w^2})(\frac{xx^2x^4}{x^3x})(\frac{yy^2}{y^2})(\frac{zz^3z}{zz})$$

Next, use the product and quotient rules to raise each base term to the sum/difference of their exponents. Leave the $\frac{1}{w^2}$ *as it is for now.*

$$= \left(\frac{1}{w^2}\right)(x^{1+2+4-3-1})(y^{1+2-2})(z^{1+3+1-1-1})$$

$$= \left(\frac{1}{w^2}\right)(x^3)(y)(z^3)$$

$$= \frac{x^3yz^3}{w^2}$$

The expression we're left with is fully simplified. However, we can manipulate it further.

Knowing that $w^0 = 1$, *the term* $\left(\frac{1}{w^2}\right)$ *can be rewritten as* $\left(\frac{w^0}{w^2}\right)$, *which is equal to* w^{0-2}, *or* w^{-2}.

Substituting this term into our final expression results in the following equivalent expression:

$w^{-2}x^3yz^3$

This serves to illustrate the way ***negative exponents*** behave in general:

$$\boldsymbol{a^{-x} = \frac{1}{a^x}}$$

Which implies the following properties:

$$a^x = \frac{1}{a^{-x}}$$

Example 5 $\frac{3^{-4}m^{-2}p^4}{nq^{-2}}$ $Express\ using\ positive\ exponents.$

Every term in our expression that is raised to a positive exponent can stay where it is. Terms raised to a negative exponent must switch from numerator to denominator or vice versa.

$$= \boldsymbol{\frac{p^4q^2}{3^4m^2n}}$$

Product/Quotient to a Power Rule

It is not uncommon to encounter expressions in parentheses raised to an exponent.

Example 6 $(xy)^4$ $Expand\ and\ simplify.$

Since there are no simplifications to make within our parentheses, our first step is to expand our expression.

$$= (xy)(xy)(xy)(xy)$$

Next, we remove the parentheses, group our terms, and use the product rule to simplify.

$$= xyxyxyxy$$

$$= xxxxyyyy$$

$$= x^{1+1+1+1}y^{1+1+1+1}$$

$$= \boldsymbol{x^4y^4}$$

Our fully simplified expression consists of the exponent being distributed across the factors of the product within the parentheses in our original expression. This illustrates the ***Product to a Power Rule***.

$$\boldsymbol{(ab)^x = a^xb^x}$$

The rule also holds true for quotients, leading to the ***Quotient to a Power Rule***:

$$\boldsymbol{\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}}$$

Combined with a negative exponent, we can also state the following:

$$\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x$$

Power to a Power Rule

It is common for the terms within parentheses to be raised to exponents as well.

Example 8 $(r^2t^2s^{-1})^3$ *Expand and simplify.*

$$= (r^2t^2s^{-1})(r^2t^2s^{-1})(r^2t^2s^{-1})$$

Again, we remove our parentheses, group our terms, and use the product rule to simplify the expression further.

$$= r^2t^2s^{-1}r^2t^2s^{-1}r^2t^2s^{-1}$$

$$= r^2r^2r^2t^2t^2t^2s^{-1}s^{-1}s^{-1}$$

$$= r^{2+2+2}t^{2+2+2}s^{(-1)+(-1)+(-1)}$$

$$= r^6t^6s^{-3}$$

*Notice that in our fully simplified expression, the exponent each factor is raised to is the product of that factor's exponent and the exponent the entire expression was raised to. This illustrates the **Power to a Power Rule**.*

$$\boldsymbol{(a^x)^y = (a^y)^x = a^{xy}}$$

Example 9 $\left(\frac{3n^2(2m^3n)^3}{12m^2n^5}\right)^3$ *Simplify.*

Following the standard order of operations, we begin by simplify the innermost set of parentheses, which involved using the power to a power rule.

$$= \left(\frac{3n^2(2^3m^{3\times3}n^3)}{12m^2n^5}\right)^3$$

$$= \left(\frac{3n^2(8m^9n^3)}{12m^2n^5}\right)^3$$

Once the innermost expression is simplified, we can open our parentheses and simplify our entire numerator, using the product rule, as well as standard multiplication.

$$= \left(\frac{(3)(8)n^{2+3}\,m^9}{12m^2n^5}\right)^3$$

$$= \left(\frac{24n^5m^9}{12m^2n^5}\right)^3$$

From here, we use the quotient rule and standard division to simplify everything within the remaining set of parentheses.

$$= (2m^7)^3$$

Our final step is using the power to a power rule again to fully simplify the expression.

$= 2^3m^{7\times3}$

$= \mathbf{8m^{21}}$

Radical Expressions/Fractional Exponents

Radicals can be thought of as the opposite operation of raising a term to an exponent. Where exponents take an argument and multiply it repeatedly, the radical operator is used in an effort to find a **root term** that can be repeatedly multiplied a certain number of times to result in the argument.

Example 10 $\sqrt{16}$ $Evaluate.$

The operator used here is the "square root", the most common radical operator. By taking the square root of a term, we're attempting to determine which positive root can be multiplied by itself – or raised to the exponent 2 *– to result in our argument.*

By observation, we can see that our argument is equal to 4^2*.*

$= \sqrt{4^2}$

Finally, our square root operator "undoes" our exponent operator, leaving us with our root term.

$= \mathbf{4}$

Note: Although it may be tempting to write ±4 as our evaluated expression since $(-4)^2 = 16$, when there are both a positive and negative root, radical operators only refer to **positive roots**.

Example 11 $\sqrt[4]{81}$ $Evaluate.$

What we're trying to determine using this radical operator is the "fourth root" (indicated by the ***index*** *of* 4 *in our expression), or the positive root term that will result in* 81 *when raised to the exponent* 4.

Again, by observation we know that $81 = 3^4$*, which allows us to evaluate our expression.*

$= \sqrt[4]{3^4}$

$= \mathbf{3}$

Example 12 $\sqrt[5]{-32}$ $Evaluate.$

Like the last two examples, we want to find a root term that, when raised to the given power of 5*, will result in our argument. Unlike the last two examples, our argument is negative. Because of this, we know a positive root will not be a possibility.*

By observation, however, we can see that $(-2)^5 = -32$*, which allows us to evaluate our expression.*

$= \sqrt[5]{(-2)^5}$

$= -2$

Examples 10, 11 and 12 illustrate the following properties of radicals:

$$\boldsymbol{\sqrt[x]{a^x} = \left(\sqrt[x]{a}\right)^x = a}$$

We can also express radicals as **fractional exponents.**

$$\boldsymbol{\sqrt[x]{a} = a^{\frac{1}{x}}}$$

Expressing radicals in this way allows us to use all of the exponent rules discussed earlier in the workshop to evaluate or simplify radical expressions.

Example 13 $\left(\sqrt[10]{36m^4}\right)^5$ $\quad Simplify.$

Our first step in evaluating this expression will be converting our radical operator to a fractional exponent.

$$= \left((36m^4)^{\frac{1}{10}}\right)^5$$

Next, we use our power to a power rule to simplify.

$$= (36m^4)^{\frac{1}{10}\times 5}$$

$$= (36m^4)^{\frac{5}{10}}$$

$$= (36m^4)^{\frac{1}{2}}$$

We now have a simplified fractional exponent. Recognizing that $36 = 6^2$ *and* $m^4 = (m^2)^2$ *allows us to simplify the expression further.*

$$= (6^2(m^2)^2)^{\frac{1}{2}}$$

$$= 6^{\frac{2}{2}}(m^2)^{\frac{2}{2}}$$

$$= \boldsymbol{6m^2}$$

We can use fractional exponents to combine radical expressions with different indexes into one expression with a shared index.

Example 14 $\sqrt[4]{4mn}\,\sqrt[3]{mn^2}$ $\quad Express\ in\ terms\ of\ a\ common\ index.$

Our first step is converting our radicals to fractional exponents.

$$= (4mn)^{\frac{1}{4}}(mn^2)^{\frac{1}{3}}$$

Next, we distribute our exponents across the terms in parentheses.

$$= 4^{\frac{1}{4}}\, m^{\frac{1}{4}}n^{\frac{1}{4}}m^{\frac{1}{3}}n^{\frac{2}{3}}$$

To add our exponent terms together, we need to bring them to a common denominator. In this case, the common denominator is 12.

$$= 4^{\frac{3}{12}}\, m^{\frac{3}{12}} n^{\frac{3}{4}} m^{\frac{4}{12}} n^{\frac{8}{12}}$$

At this point, we can use the product rule.

$$= 4^{\frac{3}{12}}\, m^{\frac{7}{12}} n^{\frac{11}{12}}$$

Finally, we can use the power to a power rule to isolate $1/12$ *as our fractional exponent and convert back to a radical expression.*

$$= (4^3 m^7 n^{11})^{\frac{1}{12}}$$

$$= (64 m^7 n^{11})^{\frac{1}{12}}$$

$$= \boldsymbol{\sqrt[12]{64 m^7 n^{11}}}$$

Rationalizing Denominators

When presented with a fraction that has a radical in the denominator, it is common practice to manipulate the expression in such a way that eliminates the radical. This is called rationalizing the denominator.

Example 15 $\frac{\sqrt{x}-8}{2\sqrt{x}}$ $\quad$ $Rationalize\ the\ denominator.$

To rationalize the expression, we want to clear the $\sqrt{x}$ *in our denominator. Squaring the entire expression would accomplish this, but – after squaring – our new expression would no longer be equal to our original expression.*

Instead, we will multiply by $\frac{\sqrt{x}}{\sqrt{x}}$*, which is the same as multiplying by 1.*

$$= \frac{\sqrt{x}-8}{2\sqrt{x}}\left(\frac{\sqrt{x}}{\sqrt{x}}\right)$$

$$= \frac{\left(\sqrt{x}\right)^2 - 8\sqrt{x}}{2\left(\sqrt{x}\right)^2}$$

$$= \boldsymbol{\frac{x-8\sqrt{x}}{2x}}$$

If we are presented with a binomial radical expression in the denominator, rationalization becomes a little bit more complicated.

Example 16 $= \frac{2}{10-\sqrt{p}}$ $\quad$ $Rationalize\ the\ denominator.$

If we multiplied our entire expression by $\frac{\sqrt{p}}{\sqrt{p}}$*, we would still have a radical in the denominator. Instead, we will multiply our numerator and denominator by the conjugate of our denominator, effectively clearing all radicals in our denominator.*

$$= \frac{2}{10-\sqrt{p}}\left(\frac{10+\sqrt{p}}{10+\sqrt{p}}\right)$$

$$= \frac{2(10)+2(\sqrt{p})}{10(10)+10(\sqrt{p})-\sqrt{p}(10)-(\sqrt{p})^2}$$

$$= \frac{\mathbf{20+2\sqrt{p}}}{\mathbf{100-p}}$$

Common Mistakes

There are a few common mistakes many students tend to make when working with exponents and radicals.

The first is how a binomial behaves when raised to an exponent. There is a tendency to just distribute the exponent across the terms in parentheses, however, this is incorrect.

$$(x+y)^n \neq x^n + y^n$$

These expressions need to be expanded by repeatedly multiplying $(x+y)$ by itself, n times.

The second mistake is very similar; some people tend to distribute a square root operator across binomials, when this is incorrect.

$$\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$$

Finally, there is a tendency to use the quotient rule incorrectly in certain situations.

$$\frac{x^2+1}{x} \neq x+1$$

The assumption is made that the x^2 term in the numerator is divided by the x term in the denominator, resulting in $x^{2-1} = x$, then the $+1$ is tacked back on. In order to perform this cancellation properly, the expression needs to be split into two fractions.

$$\frac{x^2+1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}$$